Jost Bürgi's mathematical tables of the sine function and of inverse logarithms

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Abstract

In the years of 1586 to 1592 the Swiss instrument maker and mathematician Jost Bürgi devised and documented an ingenious algorithm for efficiently and precisely calculating tables of the sine function. The manuscript *Fundamentum Astronomiæ* explaining this method and Bürgi's tables had been considered as lost, but have been rediscovered in 2013 by Menso Folkerts in the University Library of Wroclaw (Poland). In this presentation we explain and discuss Bürgi's algorithm, referred to as *Artificium* or *Kunstweg*, with the tools of modern Linear Algebra. By considering the difference table of the sine function and by using matrices and eigenvalue problems, we develop a theory of the algorithm and discuss the rate of convergence.

Contents

1. Jost Bürgi's Artificium

Jost Bürgi's Artificium algorithm is described in his mathematical text Fundamentum Astronomiæ, written between 1586 and 1592, but only rediscovered 2013 by Menso Folkerts [4,5]. Details are given by, among others, Dieter Launert [6,7], Fritz Staudacher [10] and George Szpiro [11]. The Artificium is an algorithm for calculating

$$\sin\left(\frac{j\,\frac{\pi}{2}}{n}\right), \quad j = 1, \ldots, n, \quad n > 1$$

efficiently and precisely. Bürgi's example is $n = 9$ (see p. 5); he also suggests $n = 90$, *"every degree of the right angle"*. We will use the simpler case

$$n = 3: \quad \sin(30^o) = \frac{1}{2}, \quad \sin(60^o) = \frac{\sqrt{3}}{2}, \quad \sin(90^o) = 1\,.$$

From Fundamentum Astronomiæ

As customary in 16th century astronomy, the hexagesimal number system is used. Working from right to left, the algorithm generates a table of numbers. The ones printed in red are given for clarification only, they are not carried along.

$\sin(0^o)$	0		0		0		0		0		0
$\sin(30^o)$	2911	2911	780	780	209	209	56	56	15	15	4
$\sin(60^o)$	5042	2131	1351	571	362	153	97	41	26	11	7
$\sin(90^o)$	5822	780	1560	209	418	56	112	15	30	4	8
		-780		-209		-56		-15		-4	
					...	f	e	d	c	b	a

0. Initial column: $\mathbf{a} = (a_1, a_2, \ldots, a_n)' \in \mathbb{R}^n$, (almost) arbitrary, for example, but not necessarily, multiples of the sines to be calculated, $f \cdot \sin(\frac{j\,\pi}{2\,n})$, rounded to integers ($f = 8$ in the above case).

1. Next column to the left: $\mathbf{b} = (b_1, b_2, \ldots, b_n)' =$ cumulative sum of the a_j upwards, first $b_n = \frac{a_n}{2}$, then $b_j = b_{j+1} + a_j$, $j = n-1, \ldots, 1$.

2. Further column to the left: $\mathbf{c} = (c_1, c_2, \ldots, c_n)' =$ cumulative sum of the b_j downwards, first $c_1 = b_1$, then $c_j = c_{j-1} + b_j$, $j = 2, \ldots, n$.

The sine function $y = \sin(x)$

dashed: first derivative = cos(x),   dashdot: second derivative = −sin(x)
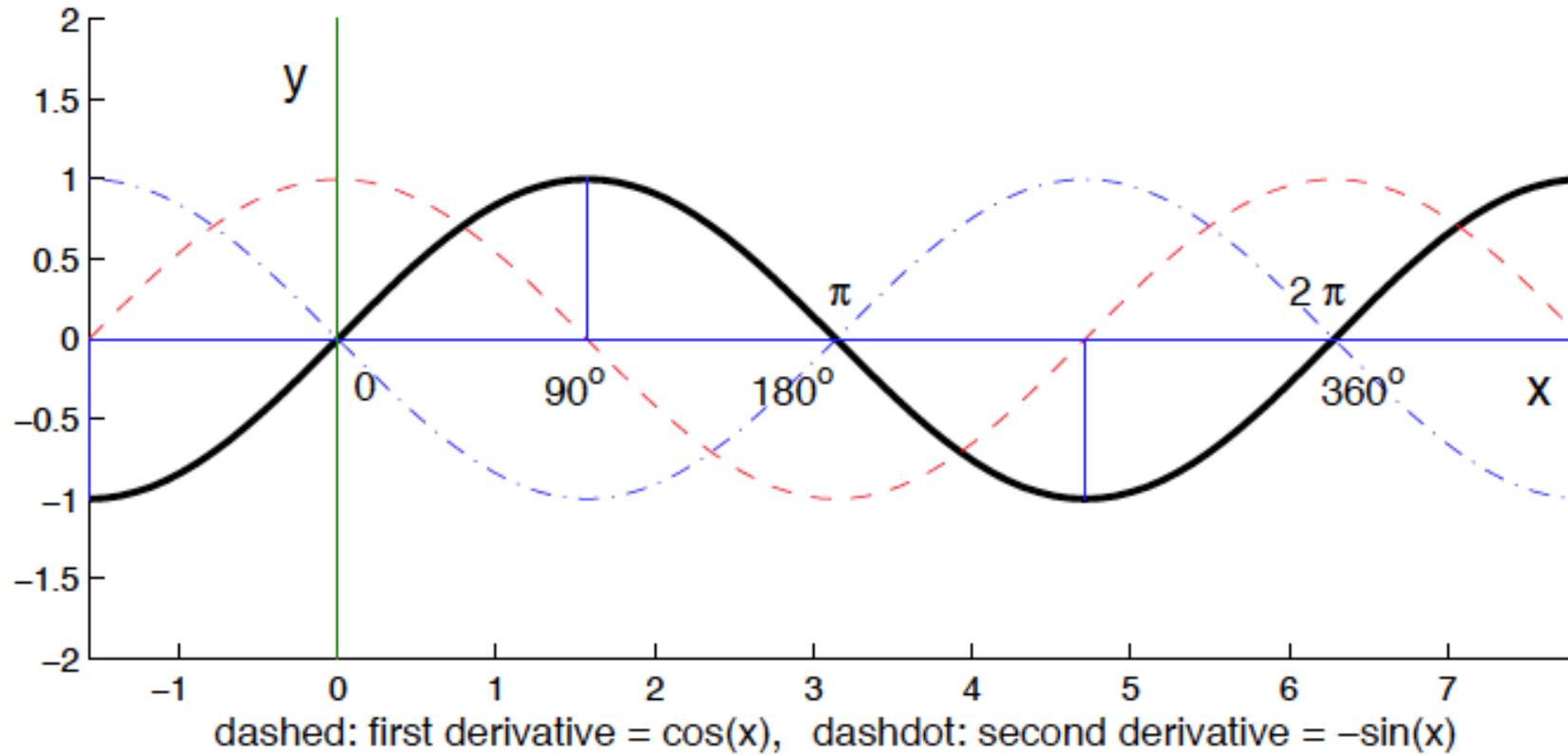
Continuation to the left

Continuation of the table to the left by repeating Steps 1 and 2. The Artificium actually is a difference table, built from right to left.

The odd columns $\mathbf{a}, \mathbf{c}, \mathbf{e}, \mathbf{g}, \ldots$, normalized by dividing them by their bottom element, approximate $\sin(\frac{j\,\pi}{2\,n})$ with increasing accuracy.

In the following table the data concerning $\sin(60^o)$ are collected:

$\ldots, c_2/c_3,\ a_2/a_3$	1351/1560	362/418	97/112	26/30	7/8
$\ldots, a_2/a_3 - \sqrt{3}/2$	2.3724e-7	3.3043e-6	4.6025e-5	6.4126e-4	8.9746e-3
Ratio to next error		13.92823	13.92855	13.93299	13.99526

The limit of the Ratio will be shown to be $7 + 4\sqrt{3} = 13.92820323$.

2. Differences

Difference tables are efficient tools for tabulating functions with equidistant arguments, e.g. the third powers $f(n) = n^3$:

n	$f(n)$	Δ_1	Δ_2	Δ_3
0	0		0	
		1		6
1	1		6	
		7		6
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		6
5	125		30	
		91		
6	216			

- In this example the third difference is constant

- The top elements and the last column must be known

- Construction of the table by cumulative summation from right to left and from top to bottom

- Additions only!

The difference table of the sine function

Prosthaphæresis (co-invented and used by Bürgi, [8])

$$\cos(\alpha) \cdot \cos(\beta) = \frac{1}{2}\Big(\cos(\alpha + \beta) + \cos(\alpha - \beta)\Big)$$

Let $\quad \alpha = \frac{\pi}{2} - \frac{y - x}{2}, \quad \beta = \frac{y + x}{2}; \quad$ this implies

$$\sin y - \sin x = 2 \, \sin\left(\frac{y - x}{2}\right) \cos\left(\frac{y + x}{2}\right).$$

Difference Table of $f(x) = \sin x$:

$f(x)$	Δ_1	Δ_2
$\sin(x - 2\,\delta)$		
	$2 \sin\delta \cdot \cos(x - \delta)$	
$\sin x$		$-4 \sin^2\delta \cdot \sin x$
	$2 \sin\delta \cdot \cos(x + \delta)$	
$\sin(x + 2\,\delta)$		

The second difference is proportional to the function value on the same line (with a negative factor)

Bürgi's main result

Remarks

- The tabulation of the cumulative sums is the inverse map of the construction of the difference table

- The initial conditions are a consequence of the symmetries of the sin- and cos-functions at $x = 0$ and $x = 90^o$

- The normalization to $\sin(90^o) = 1$ needs one division per element

Theorem 1:

For (almost) arbitrary initial columns $\mathbf{a} = (a_1, \ldots, a_n)'$ with $n > 1$, the normalized odd columns $a_k/a_n, c_k/c_n, e_k/e_n, \ldots$ converge to $\sin(k \frac{\pi/2}{n})$, $k = 1, \ldots, n$.

3. Vectors and matrices

In the case $n = 3$ of p. 4 we define $\tilde{\mathbf{a}} = (a_1, a_2, \frac{a_3}{2})' = \mathbf{H} \cdot \mathbf{a}$, where $\mathbf{H}$ generally is the diagonal matrix with the n diagonal elements $1, 1, \ldots, 1, \frac{1}{2}$. Now the first two steps of p. 6 are

$$
\mathbf{b} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tilde{\mathbf{a}}, \qquad \mathbf{c} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{\mathbf{T}} \mathbf{b} .
$$

Therefore the mapping corresponding to an Artificium step is

$$
\boxed{\mathbf{c} = \mathbf{M}\,\mathbf{a} \quad \text{with} \quad \mathbf{M} = \mathbf{T} \cdot \mathbf{T}' \cdot \mathbf{H} .}
$$

The Bürgi matrix M

The matrix $\mathbf{M} = \mathbf{T} \cdot \mathbf{T}' \cdot \mathbf{H}$ describing the Artificium mapping will be called the Bürgi Matrix. It had already been mentioned by D. Launert and A. Thom [6]. For $n = 5$ we obtain

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0.5 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 2 & 3 & 3 & 1.5 \\ 1 & 2 & 3 & 4 & 2 \\ 1 & 2 & 3 & 4 & 2.5 \end{pmatrix}.$$

The matrix T has a simple inverse

$\mathbf{T} \in \mathbb{R}^{n \times n}$ is the lower triangular matrix filled with ones. For later use we derive an alternative representation of $\mathbf{T}$. Let $\mathbf{I}$ be the unit matrix, and let $\mathbf{L}$ be the unit subdiagonal matrix,

$$\mathbf{I} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \qquad \mathbf{L} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Then we have

$$\mathbf{T} = \mathbf{I} + \mathbf{L} + \mathbf{L}^2 + \cdots + \mathbf{L}^{n-1} = (\mathbf{I} - \mathbf{L})^{-1}.$$

The Bürgi matrix M also has a simple inverse

If $\mathbf{a}^{(0)} = \mathbf{a} \in \mathbb{R}^n$ is the initial column and the further odd columns are denoted by $\mathbf{a}^{(1)} = \mathbf{c}$, $\mathbf{a}^{(2)} = \mathbf{e}$, ..., the Artificium algorithm may be written as

$$\mathbf{a}^{(j)} = \mathbf{M}\,\mathbf{a}^{(j-1)}, \quad j = 1, 2, \dots,$$

in modern Linear Algebra known as power iteration (von Mises - Geiringer). Assume $\mathbf{M}$ has only one eigenvalue, λ_1, of maximum magnitude. Then the normalized vectors $\mathbf{a}^{(j)}/\|\mathbf{a}^{(j)}\|$ converge to the corresponding eigenvector $\mathbf{v_1}$ with the property $\mathbf{M}\,\mathbf{v_1} = \mathbf{v_1}\,\lambda_1$.

For solving the eigenvalue problem we consider the inverse of $\mathbf{M}$, which is tridiagonal (use $\mathbf{T}$ of p. 14):

$$\mathbf{M}^{-1} = \mathbf{H}^{-1}\,(\mathbf{I} - \mathbf{L})'\,(\mathbf{I} - \mathbf{L}).$$

The mapping induced by $\mathbf{M}^{-1}$

For $n = 4$ we obtain

$$\mathbf{M}^{-1} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -2 & 2 \end{pmatrix}, \quad \tilde{\mathbf{M}} = \begin{pmatrix} -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 & -1 \end{pmatrix}.$$

By introducing the supplemented vector $\tilde{\mathbf{x}} = \big(0; \mathbf{x}; x_{n-1}\big)$, exactly modelling the behaviour of the sine funtion at 0 and at $\pi/2$, the mapping induced by $\mathbf{M}^{-1}$ becomes

$$\mathbf{y} = \tilde{\mathbf{M}}\,\tilde{\mathbf{x}}$$

Results

- The Artificium algorithm inverts the formation of the difference table of the sine function in the interval $[0, \frac{\pi}{2}]$ up to an unknown factor

- Bürgi takes care of this factor by normalizing the leftmost column to $\sin(\pi/2) = 1$ by dividig it by its nth element.

Power iteration

Notation: upper index (j) counts the odd columns:

Initial column $\qquad \mathbf{a}^{(0)} := \mathbf{a} \in \mathbb{R}^n$

Further odd columns $\quad \mathbf{a}^{(1)} := \mathbf{c}, \ \mathbf{a}^{(2)} := \mathbf{e}, \ \ldots$

with components $\mathbf{a}^{(j)} = \big(a_1^{(j)}, a_2^{(j)}, \ldots, a_n^{(j)}\big)'$.

The Artificium algorithm

$$\mathbf{a}^{(j)} = \mathbf{M}\,\mathbf{a}^{(j-1)}\,, \quad \mathbf{s}^{(j)} = \frac{\mathbf{a}^{(j)}}{a_n^{(j)}}\,, \quad j = 1, 2, \ldots,$$

is the well-known power iteration (R. von Mises, Hilda Geiringer, 1929).

Relevant for convergence theory: Eigenvalue problem of M.

4. Eigenvectors and Eigenvalues

The Eigenvalueproblem of $\mathbf{M}$

If the eigenvalue λ_1 of maximum magnitude is simple, the power iteration converges direction-wise to the corresponding eigenvector $\mathbf{v_1}$ satisfying $\mathbf{M}\,\mathbf{v_1} = \mathbf{v_1}\,\lambda_1$.

$\mathbf{M}^{-1}$ has the same eigenvectors as $\mathbf{M}$, but the reciprocal eigenvalues.

The following theorem may easily be proven using elementary trigonometry (p. 10) and the mapping induced by the matrix $\tilde{\mathbf{M}}$ based on $\mathbf{M}^{-1}$, representing calculation of the negative second difference.

Theorem 2. There exists a regular matrix $\mathbf{V}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M}$ is similar to $\mathbf{D}$, i.e.

$$\mathbf{M}\,\mathbf{V} = \mathbf{V}\,\mathbf{D}\,.$$

The matrix $\mathbf{V} = \left(v_{ki}\right)$ with

$$v_{ki} = \sin\left(k\,(i - \frac{1}{2})\,\frac{\pi}{n}\right), \quad k,\,i = 1,\ldots,n$$

contains n linearly independent eigenvectors of $\mathbf{M}$ as its columns (i fixed), and $\mathbf{D}$ contains the eigenvalues

$$\lambda_i = \frac{1}{4\,\sin^2\left((i - \frac{1}{2})\,\frac{\pi}{2\,n}\right)} \quad \text{with} \quad \lambda_1 > \lambda_2 > \cdots > \lambda_n$$

on its diagonal.

5. Rate of Convergence

Bürgi's normalizations for obtaining approximations x_{kj} of sines:

$$\mathbf{s}^{(j)} = \frac{\mathbf{a}^{(j)}}{a_n^{(j)}} \quad \Longrightarrow \quad s_n^{(j)} = 1$$

Bürgi's Artificium algorithm yields

$$\lim_{j \to \infty} \mathbf{s}^{(j)} = \mathbf{v}_1 \qquad \text{(first eigenvector)}$$

Norm of the error: $\qquad\qquad e^{(j)} = \left\| \mathbf{s}^{(j)} - \mathbf{v}_1 \right\|_2$

Convergence quotient: $\qquad q^{(j)} = \dfrac{e^{(j-1)}}{e^{(j)}}$

Often we have $\qquad\qquad Q := \lim_{j \to \infty} q^{(j)} = \dfrac{\lambda_1}{\lambda_2}$

Modified initial column u = a in the basis of V

According to the first equation of Theorem 2, $\mathbf{M\,V} = \mathbf{V\,D}$, the power iteration can be represented in a simpler form:

$$\mathbf{a}^{(j)} = \mathbf{M\,V\,V}^{-1}\,\mathbf{a}^{(j-1)} \quad \Rightarrow \quad \mathbf{V}^{-1}\,\mathbf{a}^{(j)} = \mathbf{D\,V}^{-1}\,\mathbf{a}^{(j-1)}\,.$$

Modified iteration vector: $\mathbf{u}^{(j)} = \mathbf{V}^{-1}\,\mathbf{a}^{(j)}$

Modified initial column: $\mathbf{u} = \mathbf{u}^{(0)} = \mathbf{V}^{-1}\,\mathbf{a}^{(0)} = \mathbf{V}^{-1}\,\mathbf{a}$

Therefore: $\mathbf{u}^{(j)} = \mathbf{D\,u}^{(j-1)}$ or $\boxed{\mathbf{u}^{(j)} = \mathbf{D}^{j}\,\mathbf{u}^{(0)}}$

The rate of convergence depends on the eigenvalues
$\lambda_i = \frac{1}{4} \big/ \sin^2\left(\left(i - \frac{1}{2}\right)\frac{\pi}{2\,n}\right)$ and on the modified initial column $\mathbf{u}$.

The eigenvalues (p. 20) satisfy $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. With $\mathbf{V}^{-1} = \frac{2}{n}\,\mathbf{V}'\,\mathbf{H}$, the rth component u_r of the modified initial column becomes $(r = 1, 2, \ldots, n)$:

$$u_r = \frac{2}{n}\,\sum_{i=1}^{n}{}' \sin\left((r - \tfrac{1}{2})\,i\,\tfrac{\pi}{n}\right)\,a_i, \quad \Sigma' : \text{last term with half weight}$$

The initial column $\mathbf{a}$ must be chosen such that $u_1 \neq 0$. The statement of p. 21 on Q holds, if also $u_2 \neq 0$.

Theorem 3

Let $r \geq 2$ be the smallest index with $u_r \neq 0$. Then $Q = \lim_{j \to \infty} q^{(j)} = \dfrac{\lambda_1}{\lambda_r}$.

6. How Bürgi could have found the Artificium of 1586, and an application of the Progress Tabulen

6.1 A possible origin of Bürgi's ideas for the sine table

In the 16th century astronomy was one of the most advanced sciences. The knowledge had been obtained by astronomers observing stars and planets through telescopes. The position of a star was described by two angles, and for investigating the motion of planets trigonometry had to be developed. It could also be used very well for navigation in the oceans.

Jost Bürgi (1552-1632) was appointed 1579 in Kassel by the Duke Wilhelm IV of Hessen as the court Watchmaker and Mechanicus. In this position he had contact with many scientists, also with astronomers.

This could also have produced the idea to generate new and better tables of the trigonometric sine function, to be used in astronomy as well as in navigation.

At that time, all data used in astronomical calculations (also lengths) were represented as numbers in the system of base 60, still used today with minutes and seconds for times and angles. In Bürgi's trigonometric tables also the values of the sine function are represented in base 60. In our discussion these data will be converted to the decimal system.

Bürgi's idea to generate a table of sines is absolutely ingenious, to be admired even today. No documents about the development of Bürgi's ideas exist, but it can be assumed that he had a few sine-values from earlier tables, and that he had the idea to repeatedly take the differences.

Example:

Angles $\alpha_j = j \cdot 30^0$, $j = 0, 1, 2, 3$:

$\Delta_{1,j} = \sin \alpha_j - \sin \alpha_{j-1}$, $\Delta_{2,j} = \Delta_{1,j+1} - \Delta_{1,j}$, $j = 1, 2, 3$

j	α_j	$\sin \alpha_j$	Δ_1	$-\Delta_2$	$-\Delta_{2,j}/\sin \alpha_j$
0	0°	0			
			0.500000		
1	30°	0.500000		0.133975	0.267950
			0.366025		
2	60°	0.866025		0.232050	0.267948
			0.133975		
3	90°	1.000000			

$$(1)$$

To keep all values positive, the sign of Δ_2 is changed. The difference of two numbers is always noted in an intermediate line. Surprisingly, $-\Delta_2$ is proportional to $\sin \alpha$! Bürgi must have observed this too, but it is not known whether he had a proof.

For a proof one could start with the prosthaphæresis used already before 1600 to simplify multiplications, since trigonometric tables existed already:

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha \cdot \cos\beta. \qquad (2)$$

Introducing the angles δ and x according to

$$\alpha = \frac{\delta}{2} - \frac{\pi}{2}, \quad \beta = \frac{\delta}{2} + x \quad \left(\frac{\pi}{2} = 90^o\right) \qquad (3)$$

yields

$$\sin(x + \delta) - \sin x = 2\sin\frac{\delta}{2} \cdot \cos\left(x + \frac{\delta}{2}\right). \qquad (4)$$

Then three lines of a generalized table (1) with difference δ become

	Δ_1	$-\Delta_2$
$\sin(x-\delta)$	$2\sin\dfrac{\delta}{2}\cdot\cos\left(x-\dfrac{\delta}{2}\right)$	
$\sin(x)$		$2\sin\dfrac{\delta}{2}\cdot 2\sin x\cdot\sin\left(\dfrac{\delta}{2}\right)$
$\sin(x+\delta)$	$2\sin\dfrac{\delta}{2}\cdot\cos\left(x+\dfrac{\delta}{2}\right)$	

$$(5)$$

Therefore

$$-\Delta_2/\sin x = 4\sin^2\frac{\delta}{2} = 2(1-\cos\delta). \tag{6}$$

The negative second difference is proportional to $\sin(x)$ with the positive factor $f = 2(1-\cos\delta)$. In the previous example we have $\delta = 30^o = \frac{\pi}{6}$ and obtain

$$f = 2\left(1-\cos\frac{\pi}{6}\right) = 2-\sqrt{3} = .26794\ 91924.$$

We all know from school that the second derivative of $\sin(x)$ is $-1 \cdot \sin(x)$. Admirably, also the second difference is proportional to $\sin(x)$, but with a factor different from -1.

At Bürgi's time the notion of the derivative was not yet known. Bürgi did not need the derivative of the sine function, but he noticed the proportionality with the second difference and invented an ingenious sine algorithm, using this discovery.

6.2 Construction of the Artificium/Kunstweg

We use the slightly extended table (1) to give an easily understandable description of the Artificium which could have been Bürgi's way.

Add $\alpha_4 = 120^o$ to table (1) and tabulate $\mathrm{round}(f \cdot \sin(\alpha)) = [f \cdot \sin(\alpha)]$ with differences, change the signs in the columns number $-\Delta_2$, $-\Delta_3$, $-\Delta_6$, $-\Delta_7, \ldots$ and add the proportional boundary elements. Choose f, e.g. $f = 1000$, and build the repeated **integer** difference table, using $\delta = \frac{\pi}{2}/n$ with $n = 3$ and δ_j with $j = 0, \ldots, n+1$, and using $\sin(\frac{\pi}{2} + \alpha) = \sin(\frac{\pi}{2} - \alpha)$.

α	$[f \cdot \sin(\alpha)]$	Δ_1	$-\Delta_2$	$-\Delta_3$	$\Delta_4 = e$	$\Delta_5 = d$	$-\Delta_6 = c$	$-\Delta_7 = b$	$\Delta_8 = a$
0°	0		0		0		0		0
		500		134		36		10	
30°	500		134		36		10		4
		366		98		26		6	
60°	866		232		62		16		2
		134		36		10		4	
90°	1000		268		72		20		8
		-134		-36		-10		-4	
120°	866		232		62		16		2

Bürgi's discoveries:

A. Starting with the **right-most** column Δ_8, the entire table can easily be reconstructed.

Notation of column vectors $\Delta_8 = \boldsymbol{a} = (a_1, \ldots, a_n)'$, $-\Delta_7 = \boldsymbol{b} = (b_1, \ldots, b_n)'$, $-\Delta_6 = \boldsymbol{c} = (c_1, \ldots, c_n)'$.

$$\text{Get } \boldsymbol{b} \text{ from } \boldsymbol{a}, \text{ bottom to top:} \tag{7}$$
$$b_n = \frac{a_n}{2}, \ b_{j-1} = b_j + a_{j-1}, \ j = n, \ldots, 2,$$

$$\text{Get } \boldsymbol{c} \text{ from } \boldsymbol{b}, \text{ top to bottom:} \tag{8}$$
$$c_1 = b_1, \ c_j = c_{j-1} + b_j, \ j = 2, \ldots, n\,.$$

Go all the way to $\Delta_{0,j} = [f \cdot \sin(\alpha_j)]$.

B. The sines are obtained by division by $\Delta_{0,n}$

$$\sin(\alpha_j) = \Delta_{0,j}/\Delta_{0,n} \quad j = 1, \ldots, n\,.$$

C. By the generation of the difference table accuracy was lost, here from 3 digits to only one.

It can be hoped that the **inverse process** will increase the accuracy. Two more steps to the left yield

$$
\begin{array}{cccc}
 & 6964 & & 1866 \\
6964 & & 1866 & \\
 & 5098 & & 1366 \\
12062 & & 3232 & \\
 & 1866 & & 500 \\
13928 & & 3732 & \\
\end{array} .
$$

Division is heavier,

$$
\frac{12062}{13928} = 0.8660\ 2527\ 28 \qquad \frac{3232}{3732} = 0.8660\ 2357\ 98.
$$

The exact value is $\sin(60^o) = \sqrt{3}/2 = 0.8660\ 2540\ 38.$

One step improves the accuracy by a factor of ~ 14. The convergence quotient is $Q = 7 + 4\sqrt{3} = 13.9282$.

This is the Artificium:

Begin with an almost arbitrary rightmost column a, e.g. an integer 1-digit approximation of the wanted sine values.

The even columns $b, d, \ldots$ are built upwards by cumulative summation of the previous column $a, c, \ldots$. The initial (lowest) element is half the initial element of the previous column. The odd columns $c, e, \ldots$ are built by cumulative sums downwards, initial (uppermost) element $=$ uppermost element of the previous column. The boundary elements printed in red need not be carried along.

6.3 Convergence theory

For investigating the convergence the algorithm needs to be formulated mathematically. For a given $n > 1$ use the angles $\alpha_j = j \cdot \delta$, $j = 1, 2, \ldots, n$, $\delta = \frac{90^o}{n} = \frac{\pi}{2n}$ are considered.

Consider the 3 right-most column vectors denoted by $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c} \in \mathbb{R}^n$; the beginning vector $\boldsymbol{a}$ must be chosen.

We now describe the rules of §6.2 by using $n \times n$ matrices. By introducing the diagonal matrix $H = \mathrm{diag}(1, 1, \ldots, 1, \frac{1}{2})$, the halving of the last component of $\boldsymbol{a}$ may be written as

$$\widetilde{\boldsymbol{a}} = H\boldsymbol{a}.$$

Then the first operation of the Artificium is

$$\boldsymbol{b}' = \left(\ldots a_{n-2} + a_{n-1} + \frac{a_n}{2}, \ a_{n-1} + \frac{a_n}{2}, \ \frac{a_n}{2} \right)'.$$

In matrix notation with $n = 3$:

$$b = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} a \; = \; T' \cdot H a \;\; \text{with} \;\; T' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Bürgi's next step:

$$c = T \cdot b = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} a \; = \; M a,$$

where

$$M = \begin{pmatrix} 1 & 1 & 0.5 \\ 1 & 2 & 1 \\ 1 & 2 & 1.5 \end{pmatrix}.$$

Matrix theory for generating M:

$$
L = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & & 1 & 0 \end{pmatrix}, \quad \ell_{jk} = \left\{ \begin{array}{ll} 1 & \text{if } j = k+1 \\ 0 & \text{otherwise} \end{array} \right\}, \ j, k = 1, \dots, n \,.
$$

For general n there follows

$$
I + L + L^2 + L^3 + \cdots = (I - L)^{-1}, \quad I = n \times n \text{ unit matrix}
$$

$$
M = (I - L)^{-1}(I - L')^{-1} H
$$

$$
M^{-1} = H^{-1}(I - L')(I - L) = H^{-1}\big(I - L' - L + \operatorname{diag}(1, 1, \dots, 1, 0)\big).
$$

Example: $n = 4$

$$M = \begin{pmatrix} 1 & 1 & 1 & 0.5 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 3 & 1.5 \\ 1 & 2 & 3 & 2 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -2 & 2 \end{pmatrix}.$$

The multiplication by M^{-1} represents the computation of the second difference.

For investigating the convergence the eigenvalues and eigenvectors of the Bürgi matrix M are needed. Due to the simple structures of M and M^{-1} explicite expressions for the eigenvalues λ_j and eigenvectors v_{jk}

exist; the corresponding matrices are denoted by D and V:

$$\lambda_j = \frac{1}{4 \sin^2 \left(\left(j - \frac{1}{2} \right) \frac{\pi}{2n} \right)} \,, \qquad\qquad D = \mathrm{diag}(\lambda_j), \qquad\qquad (9)$$

$$v_{jk} = \sin \left(\left(j - \frac{1}{2} \right) k \frac{\pi}{n} \right), \; j, k = 1, 2, \ldots, n, \;\; V = \mathrm{matrix}\,(v_{jk}). \quad (10)$$

Then, according to linear algebra

$$M \cdot V = V \cdot D, \quad D = V^{-1} \cdot M \cdot V, \qquad\qquad (11)$$

where in this case the inverse V^{-1} is given by

$$V^{-1} = \frac{2}{n}\, V^T H. \qquad\qquad (12)$$

Bürgi's algorithm starts in the rightmost column with an (almost arbitrary) first column vector a_0. We only consider the odd columns in

Bürgi's table, now denoted by $\boldsymbol{a}_0$, $\boldsymbol{a}_1, \ldots$, i.e. we iterate

$$\boldsymbol{a}_{j+1} = M\boldsymbol{a}_j, \;\; j = 0, 1, \ldots . \tag{13}$$

To investigate the convergences, we normalize after each step by division by the n-th component $a_{j,n}$ of $\boldsymbol{a}_j$,

$$\boldsymbol{x}_j = \frac{\boldsymbol{a}_j}{a_{j,n}} , \tag{14}$$

in contrast to Bürgi, who normalized only at the end.

In case of convergence (seen later) the vector $\boldsymbol{s}$ of the wanted sine values is

$$\boldsymbol{s} = \lim_{j \to \infty} \boldsymbol{x}_j . \tag{15}$$

With the error $e_j = \|x_j - s\|$ the convergence quotient Q is given by

$$Q = \lim_{j \to \infty} \frac{e_{j+1}}{e_j} \,. \tag{16}$$

Often, Q is found to be $Q = \frac{\lambda_1}{\lambda_2} = \sin^2\left(\frac{3\pi}{4n}\right) / \sin^2\left(\frac{\pi}{4n}\right)$.

A more careful investigation uses the diagonal form D of M in Equ. (11). Introducing the modified Bürgi column vector $\boldsymbol{u}_j$ by

$$\boldsymbol{u}_j = V^{-1} \boldsymbol{a}_j, \tag{17}$$

the iteration (13) becomes

$$\boldsymbol{u}_{j+1} = D\,\boldsymbol{u}_j, \ \ \boldsymbol{u}_j = D^j \boldsymbol{u}_0, \ u_{j,k} = (\lambda_k)^j\, u_{0,k}, \tag{18}$$

where $u_{0,k}$ are the components of the modified initial column;

$$u_{0,k} = \frac{2}{n} \sum_{\ell=1}^{n}{}' \sin\left(\left(k - \frac{1}{2}\right) \ell\, \frac{\pi}{n}\right) a_{0,\ell}, \ \ \Sigma'\text{: last term: half weight.} \tag{19}$$

Only the components of the initial vector $\boldsymbol{a}_0$ with index k (producing $u_{0,k} \neq 0$ in Equ. (19)) contribute to the convergence speed. Most importantly, $u_{0,1}$ must not vanish, $u_{0,1} \neq 0$. Then, the smallest index $r > 1$ with $u_{0,r} \neq 0$ determines the convergence quotient Q.

The dominant term in the sequence $\boldsymbol{a}_j$ grows as $\boldsymbol{a}_j = O((\lambda_1)^j)$. The first additional contribution grows as $O((\lambda_r)^j)$, i.e. the relative error grows as $O((\frac{\lambda_1}{\lambda_r})^j)$. The base is called the convergence quotient Q:

$$Q = \frac{\lambda_1}{\lambda_r} \ . \tag{20}$$

Example: $n = 3$

$$V = \begin{pmatrix} \dfrac{1}{2} & 1 & \dfrac{1}{2} \\[2mm] \dfrac{\sqrt{3}}{2} & 0 & -\dfrac{\sqrt{3}}{2} \\[2mm] 1 & -1 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} \dfrac{1}{3} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{3} \\[2mm] \dfrac{2}{3} & 0 & -\dfrac{1}{3} \\[2mm] \dfrac{1}{3} & -\dfrac{1}{\sqrt{3}} & \dfrac{1}{3} \end{pmatrix}$$

$$D = \begin{pmatrix} 2+\sqrt{3} & 0 & 0 \\[2mm] 0 & \dfrac{1}{2} & 0 \\[2mm] 0 & 0 & 2-\sqrt{3} \end{pmatrix} \qquad \begin{aligned} \lambda_1 &= 2+\sqrt{3} \\[2mm] \lambda_2 &= \dfrac{1}{2} \\[2mm] \lambda_3 &= 2-\sqrt{3} \end{aligned} \qquad \begin{aligned} \frac{\lambda_1}{\lambda_2} &= 7.46410\,16151 \\[2mm] \frac{\lambda_1}{\lambda_3} &= 13.92820\,32303 \end{aligned}$$

$$M = VDV^{-1} = \begin{pmatrix} 1 & 1 & \dfrac{1}{2} \\[2mm] 1 & 2 & 1 \\[2mm] 1 & 2 & \dfrac{3}{2} \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 2 & -1 & 0 \\[2mm] -1 & 2 & -1 \\[2mm] 0 & -2 & 2 \end{pmatrix}$$

Experiment with 4 Bürgi steps: $n = 3$, $j = 2$, $\alpha = j\dfrac{\pi}{2n} = 60^o$

	initial $\boldsymbol{a}_0'$	$\boldsymbol{u}_0'$	r	Q	$a_{4,j}/a_{4,n}$	error
1	$(4, 2, 8)$	$(5.155, 0, 2.845)$	3	13.928	$866/1000$	$-2.5404_{10} - 5$
2	$(2, 2, 8)$	$(4.488, -\dfrac{4}{3}, 2.179)$	2	7.464	$754/870.75$	$-1.0522_{10} - 4$
3	$(8, 14, 18)$	$(16.75, -\dfrac{2}{3}, .584)$	2	7.464	$2814/3249.395$	$-1.2709_{10} - 5$
4	$(1, 2, 2)$	$(2.155, 0, -.155)$	3	13.928	$362/418$	$3.3043_{10} - 6$
5	$(4, 2, 8)$	$(8.041, 0, -.041)$	3	13.928	$1351/1560$	$2.3724_{10} - 7$

6.4 A remark on checks in the Artificium

As mentioned earlier (Waldvogel, Elem. Math. 69 (2014), 89-117, p. 104), the unlikely, but correct entry

$$b^{431} = 1.04404\,00441\,01224, \ b = 1.0001$$

could have been used for extremely efficient checks in hand calculation.

Example:

Precalculation and check of the last 431 steps of the Progess Tabulen:

$$
\begin{array}{rll}
b^{500} & = & 1.0512\ 6847 \qquad\qquad \text{Get}\ \ 4 \cdot 10512\ 6847 \\
& & 420\ 5073\ 9 \qquad\qquad\qquad = 42050\ 7388 \\
& & 42\ 0507\ 4 \\
& & 4205\ 1 \\
& & 4\ 2 \\
& & 4 \\
\hline
b^{500} \cdot b^{431} = b^{500+431} & = & 1.0975\ 6638\ 0 \\
\text{Bürgi had tabulated } b^{931} & = & 1.0975\ 6638 \qquad\qquad \text{Correct ! Continue !} \\
\text{exact} & & 1.0975\ 6637\ 8\ 086
\end{array}
$$

6.5 A modern application of the Progress Tabulen

About 150 years after Bürgi, Leonhard Euler from Basel (1707-1783), one of the most talented scientists of the world, developed mathematics to the present status. He introduced the notions of the limit, of clearly defined functions (e.g. $\sin(x)$, $\exp(x)$), he introduced power series for functions, complex numbers, etc., etc.. His collected works of 92 volumes fill 3 shelves of a library.

These functions are common knowledge now, are tought in high-schools as of the 7th year. We will show that Bürgi's Progress Tabulen allow to cheaply compute the exponential function $\exp(x)$ and the natural logarithm $\ln(x)$ to 9-digit precision. Following the notation in modern mathematics, we will denote the natural logarithm of x by $\log(x)$. These notions were not yet defined at Bürgi's time, except that Napier had

tabulated $\log(x)$ in some intervals to about 6 digits precision (in a 10-year effort until 1616).

Most likely, Bürgi only wanted to speed up multiplications, divisions and square roots. However, the table has more applications that could have come up only 150 years after Bürgi, when Euler brought mathematics a huge step forward.

The Euler number

$$e = 2.71828\ 18284\ 59\ldots \tag{21}$$

is the basis of the natural logarithms $\log(x)$, defined by

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \tag{22}$$

With a table of natural logarithms, multiplication can be simplified no better than with Bürgi's tables. However, $\log(x)$ is now badly needed for

integration, e.g.

$$\int_1^x \frac{dt}{t} = \log(x). \tag{23}$$

The inverse function

$$y = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{24}$$

equivalent with $x = \log(y)$ is equally important in modern mathematics, it solves the differential equation $f'(x) = f(x)$.

In a few examples we show how values of $\log(x)$ and e^x can easily and quickly be obtained from Bürgi's table.

Get natural logarithms $\log(x)$ from Bürgi

Given $x \in [1, 10]$. First, find the exponent k of 1.0001 in Equ. (25), possibly by linear interpolation:

$$x = 1.0001^k = (1 + \varepsilon)^k. \tag{25}$$

In addition, the Bürgi number $B = 23027.0022$ is needed

$$(1+\varepsilon)^B = 10, \quad \varepsilon = 10^{-4}. \tag{26}$$

Taking the logarithm of (25) yields

$$\log(x) = k \cdot \log(1+\varepsilon) = k \cdot \left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \ldots \right) = k\varepsilon - k\varepsilon \cdot \frac{\varepsilon}{2} + k\varepsilon \cdot \frac{\varepsilon^2}{3}. \tag{27}$$

Examples

a) $x = 10$. According to Equs. (26) and (27) with $k = B$:

$$\log(10) = B\varepsilon - B\varepsilon \cdot \frac{\varepsilon}{2} + B\varepsilon \cdot \frac{\varepsilon^2}{3} - \ldots \tag{28}$$

$$
\begin{array}{rl}
\log(10) = & 2.3027\ 0022 \\
- & 1\ 1513\ \ 5 \\
+ & 8 \\
\hline
& 2.3025\ 8509\ \ 3 \\
\text{exact} & 2.3025\ 8509\ \ 2994
\end{array}
$$

b) $x = 2$. Find x in the table, $x = f_k = (1 + \varepsilon)^k$

k	$(1 + \varepsilon)^k$
$k_1 = 6931$	$f_1 = 1.9998\ 3634$
$k_2 = 6932$	$f_2 = 2.0000\ 3632$

Interpolation: $k = k_2 - \dfrac{f_2 - x}{f_2 - f_1} = 6932 - \dfrac{.0000\ 3632}{.0002\ 0000} = 6931.8184$

$$
\begin{aligned}
\log(2) = \quad & .6931\ 8184 \\
- \quad & \qquad 3465\ \ 9 \\
+ \quad & \qquad\qquad\quad 2 \\
\hline
& .6931\ 4718\ \ 3 \\
\text{exact} \quad & .6931\ 4718\ \ 0560
\end{aligned}
$$

c) $x = 9.5$

k	$(1+\varepsilon)^k$
$k_1 = 22514$	$f_1 = 9.4999\ 5857$
$k_2 = 22515$	$f_2 = 9.5009\ 0857$

Interpolation:

$$
k = k_1 + \frac{x - f_1}{f_2 - f_1} = 22514 + \frac{.0000\ 4143}{.0009\ 5000} = 22514.0436
$$

$$
\begin{array}{rl}
\log(9.5) = & 2.2514\ 0436 \\
- & 1\ 1257\ \ 0 \\
+ & 7 \\
\hline
& 2.2512\ 9179\ \ 7 \\
\text{exact} & 2.2512\ 9179\ \ 8606
\end{array}
$$

Get the exponential function e^x from Bürgi

Find k with $e^x = (1+\varepsilon)^k$: e^x is the k-th table entry

$$
\Longrightarrow x = k \cdot \log(1+\varepsilon) = k\Big(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \frac{\varepsilon^4}{4} + \ldots \Big); \qquad (29)
$$

solve for k

$$
k = \frac{x}{\varepsilon}\Big(1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{12} + \frac{\varepsilon^3}{24} + \ldots \Big) = \frac{x}{\varepsilon} + \frac{x}{2} - \frac{\varepsilon x}{12} + \frac{\varepsilon^2 x}{24} \ldots \quad (30)
$$

Examples

d) $x = 1$. According to (29):

$$k = 10000 + \frac{1}{2} - \frac{0.0001}{12} + \cdots = 10000.49999 \qquad (31)$$

Interpolation:

k	$(1 + \varepsilon)^k$
$k_1 = 10000$	$f_1 = 2.7181\ 4593$
$k_2 = 10001$	$f_2 = 2.7184\ 1774$

Use the middle value, rounded down due to minus sign in (31)

$$\frac{f_1 + f_2}{2} = 2.7182\ 8183$$

$$\text{exact} \qquad 2.7182\ 8182\ 8489$$

e) $x = 10$. According to (31):

$$k = 10^4 \cdot 10 + \frac{10}{2} - 10^{-4} \cdot \frac{10}{12} + 10^{-8} \cdot \frac{10}{24} - \ldots.$$

Not in the range of the Artificium, reduce N times by the Bürgi number B

$$N = \text{floor}\left(\frac{k}{B}\right) = 4; \quad k' = k - N \cdot B = 7896.9911$$

k'	$(1 + \varepsilon)^{k'}$
$k_1' = 7896$	$f_1 = 2.2024\ 2830$
$k_2' = 7897$	$f_2 = 2.2026\ 4854$

Interpolation: $f = f_2 - \varepsilon(f_2 - f_1)(k_2' - k') = f_2 - \varepsilon \cdot 2.2024 \cdot 0.0089$

$$= 2.2026 \ 4854$$
$$\text{-} \qquad 196$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$2.2026 \ 4658$$

$$e^{10} = 2.2026 \ 4658 \cdot 10^N = \ 22026.4658$$

$$\text{exact} \qquad \varepsilon^{10} = \ 22026.4657 \ 9481 \ .$$

7. Further examples

The cases $n = 3$ and $n = 9$ are the introductory example and Bürgi's example yielding $r = 3$. The case $n = 4$ is one of the many examples with $r = 2$ and rather slow convergence $Q < 9$, where integer initial columns are difficult to find or do not exist.

n=3: $\quad \mathbf{a} = (4, 7, 8)', \quad u_2 = \frac{2}{3}\left(1 \cdot a_1 + 0 \cdot a_2 - 1 \cdot \frac{a_3}{2}\right) = 0, \ r = 3 \ \Rightarrow$

$$Q_3 = \frac{\lambda_1}{\lambda_3} = \frac{\sin^2(75^o)}{\sin^2(15^o)} = 7 + 4\sqrt{3} = 13.92820$$

n=4: $\quad \mathbf{a} = (4, 7, 9, 10)', \quad u_2 = 0.20111, \quad r = 2 \quad \Rightarrow$

$$Q_2 = \frac{\lambda_1}{\lambda_2} = \frac{\sin^2(33.75^o)}{\sin^2(11.25^o)} = 8.10973$$

n=9: $\quad \mathbf{a} = (2, 4, 6, 7, 8, 9, 10, 11, 12)', \quad u_2 = 0, \quad r = 3 \quad \Rightarrow$

$$Q_3 = \frac{\lambda_1}{\lambda_3} = \frac{\sin^2(25^o)}{\sin^2(5^o)} = 23.51281$$

n=15: $\quad \mathbf{a} = (1, 2, 4, 5, 6, 7, 8, 9, 10, 10, 11, 11, 12, 12, 12)', \ u_2 = u_3 = 0,$

$$r = 4 \ \Rightarrow \quad Q_4 = \frac{\lambda_1}{\lambda_4} = \frac{\sin^2(21^o)}{\sin^2(3^o)} = 46.88760$$

These two examples considering values of n divisible by 15, $n = 15\,m$, were found by Grégoire Nicollier [9]. They are characterized by initial columns with only a few non-zero elements. The last example shows a remarkable initial column leading to $r = 6$ and $Q_6 \approx 121$, however only with irrational components (involving the golden ratio $\phi = \frac{\sin 42^o + \sin 78^o}{\sin 6^o + \sin 66^o} = 2\cos 36^o = \frac{1+\sqrt{5}}{2} = 1.618034$).

n=15 m: $\quad a_k = 1$ if $k = 2\,m$ or $k = 10\,m$ or $k = 12\,m$, $\; a_k = 0$ otherwise,

$\qquad\qquad r = 4 \Rightarrow \quad Q_4 \approx 49$, e.g. $\;Q_4 = 48.94\;$ for $\;n{=}90\;$ (Nicollier)

n=15 m: $\quad a_k = 1$ if $k = m$ or $k = 11\,m$, $\; a_k = \phi$ if $k = 7\,m$ or $k = 13\,m$,

$\qquad\qquad a_k = 0\;$ otherwise, $\quad r = 6 \Rightarrow$

$\qquad\qquad Q_6 = \dfrac{\lambda_1}{\lambda_6} \approx 121$, goes back to $Q_4 \approx 49$ after a few steps if ϕ is only approximated, e.g. by $\phi \approx \frac{13}{8}$ (Grégoire Nicollier, Sion [9])

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